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On First-Degree Multivariate Polynomial Approximation

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INTRODUCTION

In [14], G. D. Taylor enumerated all H-sets relative to first-degree multivariate polynomials; more recently, Carasso and Laurent [3, 4] and Collatz [5] introduced chains of supports in the context of a generalized exchange algorithm which converge even if Haar's condition is not satisfied. The purpose of this paper is to extend Taylor's results to this new concept of chain.

Let Q be a compact subset of $\mathbb{R}^p$ and F a subspace of $C(Q)$ spanned by $\{f_1,...,f_n\}$. Given a continuous function g on Q, the best approximation of g in F is the element f_0 of F such that

$$\|f_0 - g\| = \inf_{f \in F} \|f - g\|,$$

where

$$\|h\| = \sup_{x \in Q} |h(x)|.$$

A subset $S = \{p_1,..., p_m\}$ of Q is called a support of a subspace V of $\mathbb{R}^n$ if there exist real numbers $\lambda(p_i)$ $(i = 1,..., m + 1)$ not all zero such that

$$\sum_{i=1}^{m+1} \lambda(p_i)\,\mu(p_i) \in V,$$

where

$$\mu(p) = (f_1(p),..., f_n(p))^T.$$

A support S is said to be minimal when no proper subset of S is a support; all possible characteristic coefficients $\lambda(p_i)$ are then nonzero and lie in a space of dimension 1. If $V = \{0\}$, a minimal support is a minimal H-set

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[1, 5, 9]. As for H-sets, one can associate to a minimal support a sign pattern $e = (e_1,..., e_{m+1})$ such that

$$e_i = \text{sign}\, \lambda(p_i).$$

A sequence of minimal supports can build a regular chain as follows.

Let $S_1 = \{p_{1,i}; i = 1,..., m_1 + 1\}$ be a minimal H-set relative to F with sign pattern $e_1 = (e_{1,1},..., e_{1,m_1+1})$. The linear subspace V_1 of $\mathbb{R}^n$ spanned by the m_1 independent vectors $\mu(p_{1,i})$ $(i = 1,..., m_1)$ has the following properties:

(a) For all $a \in V_1^\perp$, one has

$$\sum_{i=1}^{n} a_i f_i(p_{1,j}) = 0 \qquad (j = 1,..., m_1 + 1);$$

(b) If W_1 is the variety of all coefficients of best approximations of any function g in F on S_1, $V_1^\perp$ is parallel to W_1.

Now, let $S_2 = \{p_{2,i};\ i = 1,..., m_{2+1}\}$ be a support of V_1. The space $V_2 = \text{span}\{u(p_{1,i}),\ i = 1,..., m_1;\ u(p_{2,i}),\ i = 1,..., m_2\}$ has dimension $m_1 + m_2$ and properties similar to those of V_1.

Repeating this process, one obtains a chain $C = (S_1,..., S_M)$ when $V_M = \mathbb{R}^n$ and if every support of one point is deleted, the chain becomes regular.

Two regular chains $C^{(1)} = (S_1^1,..., S_{M_1}^1)$ and $C^{(2)} = (S_1^2,..., S_{M_2}^2)$ are said to lie in the same class if $M_1 = M_2$ and if, for all $i = 1,..., M_1$, card $S_i^1 =$ card S_i^2 and the sign patterns associated to S_i^1 and S_i^2 are such that either $e_i^1 = e_i^2$ or $e_i^1 = -e_i^2$.

BASIC THEOREM

Let P_n^j be the space of n-variable polynomials of degree at most j and, for a real a, let $[a]$ denote the greatest integer in a.

THEOREM 1. *There exist exactly* $[n/2] + 1$ *classes of chains of* P_n^1 *composed of a single support.*

Proof. If $C = (S_1)$ is a chain of P_n^1, S_1 is to be a minimal H-set of $n + 2$ points and the result is given by Lemma 4 of [14]. ∎

If $C = (S_1,..., S_M)$ i a regular chain of P_n^1, one will call an extension of C in P_{n+i}^1 (with $i > 0$) every regular chain C^* such that there exists an injective homomorphism h_i from $\mathbb{R}^n$ to $\mathbb{R}^{n+i}$ with

$$C^* = (h_i(S_1),..., h_i(S_M), S_{M+1}^*).$$

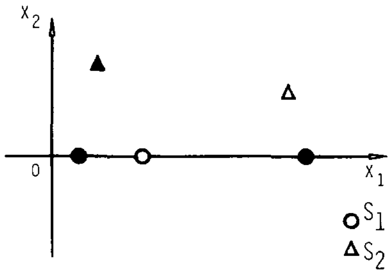

FIGURE 1

EXAMPLE. The subset $S = \{x, y, z: x < y < z\}$ of $\mathbb{R}$ is a minimal H-set and a regular chain of $P_1^1 = \text{span}\{1, x_1\}$. The set $S_1 = h_1(S) = \{{}^T(x, 0), {}^T(y, 0), {}^T(z, 0)\}$ determines a minimal H-set relative to $P_2^1 = \text{span}\{1, x_1, x_2\}$. As $V_1 = \text{span}\{{}^T(1, x, 0), {}^T(1, y, 0)\}$ has dimension 2, the set $S_2 = \{{}^T(u_1, u_2), {}^T(v_1, v_2)$: $u_2, v_2 \neq 0\}$ completes S_1 to build a regular chain of P_2^1; $C^{(1)} = (S_1, S_2)$ is an extension of (S) in P_2^1 (see Figs. 1, 2).

The H-set

$$T_1 = h_2(S) = \{{}^T(x, 0, 0), {}^T(y, 0, 0), {}^T(z, 0, 0)\}$$

relative to P_3^1 joined with

$$T_2 = \{{}^T(u_1, u_2, u_3), {}^T(v_1, v_2, v_3), {}^T(w_1, w_2, w_3)\}$$

such that the x_1 axis cuts the plane determined by u, v and w in a single point forms a regular chain $C^{(2)} = (T_1, T_2)$ which is an extension of (S) in P_3^1 (see Figs. 3, 4).

If

$$R_1 = h_1(S_1) = \{{}^T(x, 0, 0), {}^T(y, 0, 0), {}^T(z, 0, 0)\}$$

and

$$R_2 = h_1(S_2) = \{{}^T(u_1, v_1, 0), {}^T(u_2, v_2, 0): u_2, v_2 \neq 0\},$$

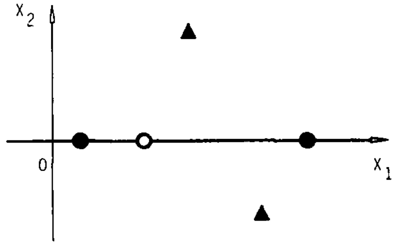

FIGURE 2

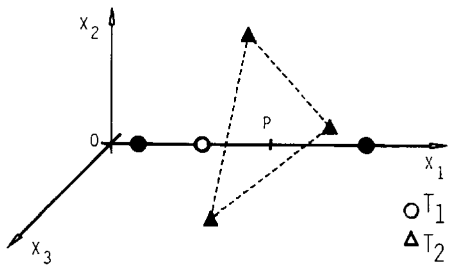

FIGURE 3

then

$$C^{(3)} = (R_1, R_2, R_3)$$

with

$$R_3 = \{{}^T(a_1, a_2, a_3), {}^T(b_1, b_2, b_3): a_3, b_3 \neq 0\}$$

is a regular chain of P_3^1 and an extension of $C^{(1)}$ in P_3^1 (see Figs. 5, 6).

THEOREM 2. *If C is a regular chain of P_n^1, there exist $[(i+1)/2]+1$ classes of extensions of C in P_{n+i}^1.*

Proof. Let $C = (S_1, ..., S_M)$ be a regular chain of P_n^1, and $C^* = (h_i(S_1), ..., h_i(S_M), S_{M+1}^*)$ be an extension of C in P_{n+i}^1. One has $\dim V_M = n$ and $\dim V_M^* = n+i$ so that S_{M+1}^* must have $i+1$ points and its associated sign pattern may be chosen in $[(i+1)/2]+1$ different ways. ∎

EXAMPLE. Let $C^{(i)}$ $(i = 1, 2, 3)$ be defined as above. For all i, the first associated sign patterns are $e_i^{(i)} = (1, -1, 1)$. For the extension $C^{(1)}$, if u_2 and v_2 are chosen such that $u_2 \cdot v_2 > 0$ (Fig. 1), $e_2^{(1)} = (1, -1)$ and otherwise (Fig. 2) $e_2^{(1)} = (1, 1)$. Concerning $C^{(2)}$, $e_2^{(2)}$ will be determined by the position of the cutting point p of the x_1 axis in the plane u, v, w. Indeed, if p is inside

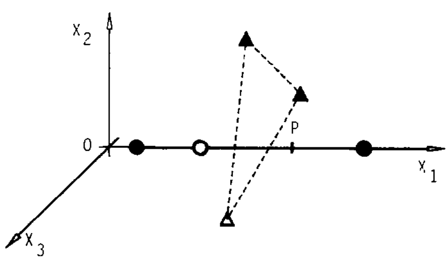

FIGURE 4

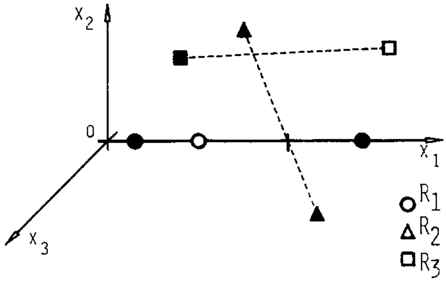

FIGURE 5

the triangle u, v, w (Fig. 3), $e_2^{(2)} = (1, 1, 1)$ and otherwise (Fig. 4) $e_2^{(2)} = (1, 1, -1)$. If $C^{(3)}$ is an extension of a $C^{(2)}$ (Figs. 1, 2), and if $b_3 \cdot a_3 > 0$ (Fig. 5), $e_3^{(3)} = (1, -1)$, and if not (Fig. 6) $e_3^{(3)} = (1, 1)$.

THEOREM 3. *If $c(n)$ represents the number of classes of chains of P_n^1,*

$$c(n) = [n/2] + 1 + \sum_{i=1}^{n-1} ([(i+1)/2] + 1)\, c(n-i). \tag{1}$$

Proof. The first term is given by Theorem 1 and the sum is induced by Theorem 2. ∎

THEOREM 4. *If $n \geqslant 5$,*

$$c(n) = 3c(n-1) + c(n-2) - 2c(n-3). \tag{2}$$

Proof. From Theorem 3, if $n \geqslant 5$, one has

$$c(n) - c(n-2) = 1 + 2c(n-2) + 2c(n-1) + \sum_{i=1}^{n-3} c(i), \tag{3}$$

$$c(n-1) - c(n-3) = 1 + 2c(n-3) + 2c(n-2) + \sum_{i=1}^{n-4} c(i). \tag{4}$$

(2) is obtained by substracting (4) from (3). ∎

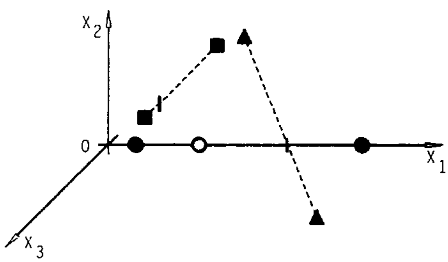

FIGURE 6

The result (2) leads to the construction of Table 1, which shows $c(n)$ with $n \leqslant 10$, given the initial values

$$c(1) = 1, \qquad c(2) = 4, \qquad c(3) = 12. \tag{5}$$

Finally, one gets an explicit form for $c(n)$.

THEOREM 5.

$$c(n) = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n, \tag{6}$$

where the rounded values of the parameters are

$$\begin{aligned} \alpha_1 &= -0.106464, & r_1 &= 0.745898, \\ \alpha_2 &= 0.203653, & r_2 &= -0.860806, \\ \alpha_3 &= 0.402810, & r_3 &= 3.114908. \end{aligned}$$

Proof. r_1, r_2, r_3 are the roots of the characteristic equation derived from the recurrent relation (2) and α_1, α_2, α_3 are fitted to the initial conditions (5). ∎

If $(a)_R$ denotes the nearest integer to the real a, one obtains a simpler form for $c(n)$.

THEOREM 6.

$$c(n) = (\alpha_3 r_3^n)_R. \tag{7}$$

Proof. If $n \geqslant 1$, $|\alpha_1 r_1^n + \alpha_2 r_2^n| < 0.26$, so that (6) leads directly to (7). ∎

TABLE I

Number of Classes of Chains of P_n^1 According to the Dimension n

n	$c(n)$
1	1
2	4
3	12
4	38
5	118
6	368
7	1,146
8	3,570
9	1,120
10	34,638

COROLLARY. *If G is a $n+1$ dimensional subspace of $c(Q)$, the number of classes of chains of G is not greater than $c(n)$.*

Proof. It is quite easily seen that P_n^1 possesses the maximum number of classes of chains among all spaces of dimension $(n+1)$. Indeed, every possible case for every support has been considered in the preceding counting.

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